

# Fourier Analysis and Holographic Representations of $1D$ and $2D$ Signals

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**Abstract.** In this paper, we focus on Fourier analysis and holographic transforms for signal representation. For instance, in the case of image processing, the holographic representation has the property that an arbitrary portion of the transformed image enables reconstruction of the whole image with details missing. We focus on holographic representation defined through the Fourier Transforms. Thus, We firstly review some results in Fourier transform and Fourier series. Next, we review the Discrete Holographic Fourier Transform (**DHFT**) for image representation. Then, we describe the contributions of our work. We show a simple scheme for progressive transmission based on the DHFT. Next, we propose the Continuous Holographic Fourier Transform (**CHFT**) and discuss some theoretical aspects of it for  $1D$  signals.

## 1 Introduction

The techniques for signal representation composes a class of methods used to characterize the information therein [12, 16]. In the area of digital image processing, a classical approach for image representation is based on unitary matrices that define the so called *image transforms*. The Discrete Fourier Transform (**DFT**) is the most known example in this class.

Stochastic models describe an image as a member of an ensemble, which can be characterized by its mean and covariance functions. This allows development of algorithms that are useful for an entire class or an ensemble of images rather than for a single image. Fall in this category the covariance models, the  $1 - D$  models (an image is considered a one-dimensional random signal that appears at the output of a raster scanner) and  $2 - D$  models (Causal, Semicausal and Noncausal) [16].

Multiresolution representations, such Pyramid [17] and Wavelets [20], compose another important class in this field. Finally, for image understanding models, an image can be considered as a composition of several objects detected by segmentation and computer vision techniques [12, 16].

In this paper we focus on the first category of image representation methods. We firstly review some results in Fourier transform and Fourier series and show some connections between them (section 2). Next, in section 3, we describe the discrete Holographic Fourier Transform (**DHFT**) proposed in [5], which is based on the DFT. The holographic representation of an image has the property that an arbitrary portion of the holographic representation enables reconstruction of the whole image with details missing. This is an interesting property that motivates works using holographic representations for image compression [2], multimedia systems [11], watermarking (see

[5] and references therein) and representations of line images [13]. Besides the holographic representation based on the DFT there are also approaches that use subsampling methods [4].

In sections 4 and 5, we show our contributions in this field. We propose the Continuous Holographic Fourier Transform (section 4) and discuss some theoretical and numerical aspect of it. Then, in section 5, we develop a simple scheme for progressive transmission of images through the DHFT. Section 6 presents our conclusions and some comments. The Appendix reviews interesting results in discrete Fourier analysis. An extended version of this work can be found in [1]

## 2 Fourier Analysis Revised

In this section we review some important results about continuous Fourier Analysis. A special attention will be given to the connections between the Fourier Transform and Fourier Series representation of a signal. Thus, let us begin with usual definitions in this area. For simplicity, we restrict our discussion to the one dimensional case. Generalizations to higher-dimensional cases are straightforward. In this section our presentation follows the reference [7].

Throughout this text, all functions  $f$  are defined on the real line, with values in the complex domain  $C$ ,  $f : \mathbb{R} \rightarrow C$ . Besides, they are assumed to be piecewise continuous and having the integral:

$$\int_{-\infty}^{\infty} |f(x)|^p dx, \quad (1)$$

finite. The set of such functions is denoted by  $L^p(\mathbb{R})$ . In this set, the  $p$ -norm and the inner product are well defined by:

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p dx \right\}^{1/p}, \quad 1 \leq p < \infty, \quad (2)$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx, \quad f, g \in L^2(\mathbb{R}). \quad (3)$$

*Definition 1:* Given a function  $f \in L^1(\mathbb{R})$ , its Fourier Transform is defined by:

$$\widehat{f}(\omega) = (Ff)(\omega) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi j\omega x) dx. \quad (4)$$

*Definition 2:* Given the Fourier transform  $\widehat{f}$ , the transformed function  $f$  can be obtained by the Inverse Fourier Transform, given as follows:

$$f(x) = (F^{-1}\widehat{f})(x) = \int_{-\infty}^{\infty} \widehat{f}(\omega) \exp(2\pi j\omega x) d\omega. \quad (5)$$

*Definition 3:* Let  $f$  and  $g$  be functions in  $L^1(\mathbb{R})$ . Then the *convolution* of  $f$  and  $g$  is also an  $L^1(\mathbb{R})$  function  $h$  which is defined by:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy. \quad (6)$$

*Theorem 1:* Any function  $f \in L^2(-L/2, L/2)$  has a Fourier series representation given by:

$$f(x) = \sum_{n=-\infty}^{+\infty} a_n \exp\left(2j\pi x \frac{n}{L}\right). \quad (7)$$

Dem: See [9].

Among the properties of the Fourier Transform, the following ones have a special place for signal processing techniques, according to our discussion presented in [1].

*Property 1:* If the derivative  $df/dx$  of  $f$  exists and is in  $L^1(\mathbb{R})$ , then:

$$\left[ F \left( \frac{df}{dx} \right) \right] (\omega) = \widehat{f}'(\omega) = j\omega \widehat{f}. \quad (8)$$

*Property 2:*  $\widehat{f} \rightarrow 0$ , as  $\omega \rightarrow \infty$  or  $\omega \rightarrow -\infty$ .

For the proofs, see [7], pp. 25.

*Property 3 (Convolution Theorem):* The Fourier transform of the convolution of two functions is the product of their Fourier transforms, that is:

$$h(x) = (f * g)(x) \Leftrightarrow \widehat{h}(\omega) = \widehat{f}(\omega) \widehat{g}(\omega). \quad (9)$$

Dem: See [9].

An important aspect for signal (and image) processing is the relationship between the coefficients  $a_n$  in the series (7) and the Fourier transform defined by the integral (4), in the case of a function  $f \in L^2(-L/2, L/2)$ . In order to find this relationship, let us remember that, from the orthogonality of the functions  $\{\exp(2j\pi x \frac{n}{L}), n \in \mathbb{Z}\}$ , it is easy to show what the coefficients  $a_n$  can be obtained by the expression:

$$a_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \exp\left(-2j\pi x \frac{n}{L}\right) dx. \quad (10)$$

Now, we must observe that this expression is equivalent to:

$$\frac{1}{L} \widehat{f}\left(\frac{n}{L}\right) = \frac{1}{L} \int_{-\infty}^{\infty} f(x) \exp\left(-2j\pi x \frac{n}{L}\right) dx, \quad (11)$$

once  $f \in L^2(-L/2, L/2)$ .

Thus,

$$a_n = \frac{1}{L} \widehat{f}\left(\frac{n}{L}\right). \quad (12)$$

This result shows the connections between Fourier series and the Fourier Transform. It is used in [1] to discussion Fourier analysis in the context of signal processing.

### 3 Discrete Holographic Fourier Transform

Optical holography technology uses interference and diffraction of light to record and reproduce 3D information of an optical wavefront [3, 19]. This technology has been applied for 3D displays [18] and data storage devices [15]. From an arbitrary portion of an optical hologram that encodes a scene, a low quality version of the entire scene can be reconstructed. The quality of the recovered signal depends only on the size of the hologram portion used, but not of the place from where it was taken [8]. The holographic representation of images tries to mimic such property through image transforms. In this section, we summarize the results presented in [5].

The Discrete Holographic Fourier Transform (**DHFT**) of an image  $I = I(x, y)$  is defined as [5]:

$$H(u, v) = F^{-1} \left[ I(x, y) e^{2j\pi P(x, y)} \right], \quad (13)$$

where  $P(x, y)$  is a random phase image so that  $E[P(x, y) P(\bar{x}, \bar{y})] = 0$  for  $(x, y) \neq (\bar{x}, \bar{y})$  and  $P(x, y)$  is a random variable uniformly distributed over  $[-1, 1]$ . By holographic representation, we mean that from an image portion  $H^c(u, v)$  cropped from anywhere in the complex image  $H(u, v)$ , we can, by 2D Fourier transformation, get a version of  $I(x, y)$  so that the degradation is proportional to the size of  $H^c(u, v)$ . Clearly, if  $H(u, v)$  is available we get back  $I(x, y)$  as the amplitude of its 2D Fourier transform, and the same image recovery process will be applied over cropped parts of  $H(u, v)$ . Certainly, we can take a portion of an unitary transform and perform the same process. However the result will be strong dependent from the place from where the portion was cropped.

In order to demonstrate that the image transform defined in expression (13) has the holographic property, let us follow the development found in [5] and consider the 1D discrete version of the above proposed representation method. The one-dimensional DHFT is:

$$H(u) = \sum_{k=0}^{M-1} I(k) e^{j2\pi P(k)} \frac{1}{\sqrt{M}} e^{j\frac{2\pi}{M}uk}, \quad (14)$$

where  $\{P(k)\}$  is a set of independent and identically distributed random numbers uniformly distributed over  $[0, 1]$ . We will represent the process of cropping a portion of  $H(u)$  by multiplying it with a window function  $W(u)$ , given by:

$$W(u) = \begin{cases} 1, & u \in [a, a + (L - 1)] \\ 0, & u \notin [a, a + (L - 1)] \end{cases}, \quad (15)$$

where  $a \in \{0, 1, \dots, M - L\}$  for simplicity. Now, we shall consider the question: what can be recovered from  $H^c(u) = H(u) \cdot W(u)$  by the one-dimensional Fourier transform. To answer this question we call  $I_W(r)$  the obtained signal. Thus:

$$\begin{aligned} I_w(r) &= \sum_{u=0}^{M-1} H(u) W(u) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}ur} = \\ &= \sum_{u=0}^{M-1} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi P(k)} \frac{1}{\sqrt{M}} e^{j\frac{2\pi}{M}uk} \right) W(u) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}ur} = \\ &= \frac{1}{M} \sum_{u=0}^{M-1} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi P(k)} e^{j\frac{2\pi}{M}u(k-r)} W(u) \right) = \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \left( \sum_{u=0}^{M-1} e^{-j\frac{2\pi}{M}u(r-k)} W(u) \right) I(k) e^{j2\pi P(k)}. \end{aligned} \quad (16)$$

Let us define  $g(r, k)$  as follows:

$$g(r, k) = \sum_{u=0}^{M-1} e^{-j\frac{2\pi}{M}u(r-k)} W(u) = \sum_{u=a}^{a+(L-1)} e^{-j\frac{2\pi}{M}u(r-k)}. \quad (17)$$

Thus, by changing variable ( $\alpha = u - a$ ), we find:

$$g(r, k) = e^{-j\frac{2\pi}{M}(r-k)a} \sum_{\alpha=0}^{L-1} e^{-j\frac{2\pi}{M}\alpha(r-k)}. \quad (18)$$

But, from the Property A2, in the Appendix:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = L e^{-j(L-1)\pi x} \frac{\text{sinc} L\pi x}{\text{sinc} \pi x}. \quad (19)$$

Through expressions (19) and (18) we can rewrite equation (16) as:

$$I_W(r) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j\frac{2\pi}{M}(r-k)a} e^{-j(L-1)\pi \frac{(r-k)}{M}} \frac{L \text{sinc} \frac{L\pi(r-k)}{M}}{\text{sinc} \frac{\pi(r-k)}{M}} I(k) e^{j2\pi P(k)}.$$

Thus, we finally have:

$$I_w(r) = \sum_{k=0}^{M-1} e^{-j\frac{2\pi}{M}ra} e^{j\frac{2\pi}{M}ka} e^{-j(L-1)\frac{\pi}{M}r} e^{j(L-1)\frac{\pi}{M}k} \frac{L \text{sinc} \frac{L\pi(r-k)}{M}}{M \text{sinc} \frac{\pi(r-k)}{M}} I(k) e^{j2\pi P(k)} =$$

$$\begin{aligned}
&= e^{-j\frac{2\pi}{M}\left[a+\frac{(L-1)}{2}\right]r} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi\left[P(k)+\frac{k}{M}\left(a+\frac{L-1}{2}\right)\right]} \frac{L \operatorname{sinc} \frac{L\pi(r-k)}{M}}{M \operatorname{sinc} \frac{\pi(r-k)}{M}} \right) = \\
&= e^{-j\frac{2\pi}{M}\left[a+\frac{(L-1)}{2}\right]r} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi\tilde{P}_a(k)} \frac{L \operatorname{sinc} \frac{L\pi(r-k)}{M}}{M \operatorname{sinc} \frac{\pi(r-k)}{M}} \right). \tag{20}
\end{aligned}$$

Let us define:

$$\phi_L(r-k) := \frac{L \operatorname{sinc} \frac{L\pi(r-k)}{M}}{M \operatorname{sinc} \frac{\pi(r-k)}{M}}. \tag{21}$$

Thus, we have:

$$I_W(r) = e^{-j\frac{2\pi}{M}\left[a+\frac{(L-1)}{2}\right]r} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi\tilde{P}_a(k)} \phi_L(r-k) \right). \tag{22}$$

Firstly, we should verify the result when  $L = M$ , and consequently,  $a = 0$ . In this case, expression (20) becomes:

$$I_W(r) = e^{-j\frac{2\pi}{M}\left(\frac{M-1}{2}\right)r} \sum_{k=0}^{M-1} I(k) e^{j2\pi\left[P(k)+\frac{k(M-1)}{2M}\right]} \phi_L(r-k). \tag{23}$$

Besides, from the definition of  $\phi_L$  we can see that, when  $L = M$ , we have:

$$\phi_L(r-k) = \begin{cases} 1, & r-k = 0, \\ 0, & r-k \neq 0, \end{cases} \tag{24}$$

because  $0 \leq r, k \leq M-1$ . Finally, by using the result (24) in expression (23) we find:

$$\begin{aligned}
I_W(r) &= e^{-j\frac{2\pi}{M}\left(\frac{M-1}{2}\right)r} \sum_{k=0}^{M-1} I(k) e^{j2\pi\left[P(k)+\frac{k(M-1)}{2M}\right]} \phi_L(r-k) = \\
&= e^{-j\frac{2\pi}{M}\left(\frac{M-1}{2}\right)r} I(r) e^{j2\pi\left[P(r)+\frac{r(M-1)}{2M}\right]} = \\
&= e^{-j\frac{2\pi}{M}\left(\frac{M+1}{2}\right)r} e^{j\frac{2\pi}{M}\left(\frac{M+1}{2}\right)r} I(r) e^{2\pi P(r)} = \\
&= I(r) e^{j2\pi P(r)}. \tag{25}
\end{aligned}$$

So, as expected, the whole image is recovered as the amplitude of the above result. In order to know how good will be the approximation of the original image generated when  $L < M$ , we can perform a statistical analysis of the recovered complex image.

Following [5], we will assume that  $I(k) = I_0$  constant:

$$\begin{aligned}
I_W(r) &= e^{-j\frac{2\pi}{M}\left[a+\frac{(L-1)}{2}\right]r} \left( \sum_{k=0}^{M-1} I(k) e^{j2\pi\left[P(k)+\frac{k}{M}\left(a+\frac{L-1}{2}\right)\right]} \phi_L(r-k) \right) \\
&= I_0 \sum_{k=0}^{M-1} e^{j2\pi\left[P(k)+\frac{(k-r)}{M}\left(a+\frac{L-1}{2}\right)\right]} \phi_L(r-k) = \\
&= I_0 \sum_{k=0}^{M-1} e^{j2\pi\tilde{P}_a(k-r)} \phi_L(r-k).
\end{aligned}$$

Henceforth:

$$\begin{aligned}
|I_W(r)|^2 &= I_W(r) I_W(r)^* = \\
&= |I_0|^2 \left( \sum_{k=0}^{M-1} e^{j2\pi\tilde{P}_a(k-r)} \phi_L(r-k) \right) \left( \sum_{l=0}^{M-1} e^{-j2\pi\tilde{P}_a(l-r)} \phi_L(r-l) \right) = \\
&= |I_0|^2 \left\{ \sum_{k=0}^{M-1} \phi_L^2(r-k) + \sum_{k \neq l} \sum_{k,l=0}^{M-1} e^{j2\pi\tilde{P}_a(k-r)} \phi_L(r-k) e^{-j2\pi\tilde{P}_a(l-r)} \phi_L(r-l) \right\} = \\
&= |I_0|^2 \left\{ \sum_{k=0}^{M-1} \phi_L^2(r-k) + \sum_{k \neq l} \sum_{k,l=0}^{M-1} e^{j2\pi[\tilde{P}_a(k-r)-\tilde{P}_a(l-r)]} \phi_L(r-k) \phi_L(r-l) \right\}.
\end{aligned}$$

To answer how good an estimate is  $|I_W(r)|$  for the value  $I_0$ , it is used in [5] the following result:

*Lemma 1:* Let  $\theta_0, \theta_1, \dots, \theta_{M-1}$  be independent and identically distributed random numbers uniformly chosen from  $[0, 1]$ , and the random variable:

$$V = \sum_{k=0}^{M-1} e^{j2\pi\theta_k} \varphi(k),$$

where  $\varphi(k)$  is a sequence of real numbers. Thus, we have the following statistics for  $V$ :

$$E(V) = 0, \quad E(|V|^2) = \sum_{k=0}^{M-1} \varphi^2(k), \quad (26)$$

$$\sigma(|V|^2) = \sqrt{\sum_{k \neq l} \sum_{k,l=0}^{M-1} \varphi^2(k) \varphi^2(l)}. \quad (27)$$

As  $I_W(r)$  is of the form  $\sum_{k=0}^{M-1} e^{j2\pi\theta_k} \varphi(k)$ , it follows from this Lemma that:

$$\begin{aligned}
E[|I_W(r)|^2] &= |I_0|^2 \sum_{k=0}^{M-1} \phi_L^2(r-k), \\
\sigma[|I_W(r)|^2] &= |I_0|^2 \sqrt{\sum_{k \neq l} \sum_{k,l=0}^{M-1} \phi_L^2(r-k) \phi_L^2(r-l)}.
\end{aligned}$$

If  $L = M$ , expression (24) implies that:

$$E[|I_W(r)|^2] = |I_0|^2, \quad \sigma[|I_W(r)|^2] = 0,$$

as expected from the result (25).

Supposing  $L < M$ , it follows:

$$\begin{aligned}
E[|I_W(r)|^2] &= |I_0|^2 \sum_{k=0}^{M-1} \phi_L^2(r-k) = \\
&= |I_0|^2 \sum_{k=0}^{M-1} \frac{L^2 \text{sinc}^2 \frac{L\pi(r-k)}{M}}{M^2 \text{sinc}^2 \frac{\pi(r-k)}{M}} =
\end{aligned}$$

$$= \frac{L^2}{M^2} |I_0|^2 \sum_{k=0}^{M-1} \frac{\text{sinc}^2 \frac{L\pi(r-k)}{M}}{\text{sinc}^2 \frac{\pi(r-k)}{M}}. \quad (28)$$

Simplifying the summation:

$$\sum_{k=0}^{M-1} \frac{\text{sinc}^2 \frac{L\pi(r-k)}{M}}{\text{sinc}^2 \frac{\pi(r-k)}{M}} = \frac{M}{L},$$

so, returning to expression (28) we find:

$$E \left[ |I_W(r)|^2 \right] = \frac{L^2}{M^2} I_0^2 \sum_{k=0}^{M-1} \frac{\text{sinc}^2 \frac{L\pi(r-k)}{M}}{\text{sinc}^2 \frac{\pi(r-k)}{M}} = \frac{L}{M} I_0^2,$$

$$E \left[ |I_W(r)|^2 \right] = \frac{L}{M} I_0^2.$$

So the image recovered is an approximation of the original image multiplied by the factor  $\sqrt{\frac{L}{M}}$ .

In the previous demonstrations we recovered the original image from a window in the holographic domain extended by zeros. We consider now recovering the image from the window without extending it. The Fourier transform of the cropped sequence  $H(u)$ ,  $u \in [a, a + (L - 1)]$ , is:

$$I_T(r) = FT[H(u)] = \sum_{k=0}^{L-1} H(k+a) \frac{1}{\sqrt{L}} e^{-j\frac{2\pi}{L}rk}. \quad (29)$$

Defining the extended sequence  $\tilde{H}(u)$ ,  $u \in [0, M - 1]$  and its Fourier transform as below:

$$\tilde{H}(k) = \begin{cases} H(k+a), & k \in [0, L-1], \\ 0, & k \in [L, M-L], \end{cases}$$

$$I_{WS}(v) = FT[\tilde{H}(u)] = \sum_{u=0}^{M-1} \tilde{H}(u) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}vu} =$$

$$= \sum_{k=0}^{L-1} H(k+a) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}vk}. \quad (30)$$

Comparing the two expressions (29) and (30) it can be straightforward noticed that:

$$I_{WS}(v) = \sqrt{\frac{L}{M}} I_T\left(\frac{L}{M}v\right), \quad v = 0, \quad \frac{L}{M}, \quad \frac{2L}{M}, \quad \dots, \quad \frac{(L-1)L}{M}. \quad (31)$$

We must also observe that:

$$I_w(r) = \sum_{u=a}^{a+L-1} H(u) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}ur}. \quad (32)$$

Hence, by changing  $k = u - a$ , we have

$$I_w(r) = \sum_{k=0}^{L-1} H(k+a) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}(k+a)r} =$$

$$= \left( \sum_{k=0}^{L-1} H(k+a) \frac{1}{\sqrt{M}} e^{-j\frac{2\pi}{M}kr} \right) e^{-j\frac{2\pi}{M}ar} = I_{WS}(r) \cdot e^{-j\frac{2\pi}{M}ar}. \quad (33)$$

$$I_{WS}(r) = I_W(r) \cdot e^{j\frac{2\pi}{M}ar}. \quad (34)$$

Substituting this result in (31), we have:

$$I_T\left(\frac{L}{M}v\right) = \sqrt{\frac{M}{L}} I_W(v) e^{j\frac{2\pi}{M}av}. \quad (35)$$

Considering  $\tilde{r} = \frac{L}{M}v$ , follows:

$$I_T(\tilde{r}) = \sqrt{\frac{M}{L}} I_W\left(\frac{M}{L}\tilde{r}\right) e^{j\frac{2\pi}{L}a\tilde{r}}. \quad (36)$$

Since the image is the absolute value of the holographic representation, it follows:

$$|I_T(\tilde{r})| = \sqrt{\frac{M}{L}} \left| I_W\left(\frac{M}{L}\tilde{r}\right) \right|. \quad (37)$$

Which proves that, being  $|I_T(\tilde{r})|$  a subsampling of  $\sqrt{\frac{M}{L}} |I_W(\frac{M}{L}\tilde{r})|$ , it is a good estimate of  $I_0$  (multiplying  $|I_T(\tilde{r})|$  by  $\sqrt{\frac{L}{M}}$ ), in case of  $I(k)$  constant.

#### 4 Continuous Holographic Fourier Transform

In this section we extend some developments of section 3 for continuous signals. Therefore, we start with a definition for the Continuous Fourier Holographic Transform (**CHFT**):

$$H(\omega) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi j\omega x) \exp(2\pi jP(x)) dx, \quad (38)$$

where  $P(x)$  is a random phase. Following the development of section 3, we will also consider a window in the Fourier domain and check the quality of the recovered signal. Before this, we shall arrange the terms of the integral in the following form:

$$H(\omega) = \int_{-\infty}^{\infty} \exp(-2\pi j\omega x) [f(x) \exp(2\pi jP(x))] dx. \quad (39)$$

We can think about the Fourier transform of the signal  $h(x) = f(x) \exp(2\pi jP(x))$ , which is well defined if  $f \in L^2$ . Now, we use a window function (low-pass filter) given by expression:

$$W(\omega) = \begin{cases} 1, & -k \leq \omega \leq k, \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

The filtering of signal  $h$  by  $W$  can be represented, in the Fourier domain by:

$$G(\omega) = W(\omega) H(\omega), \quad (41)$$

where  $G(\omega)$  is the Fourier transform of the output signal and  $H$  is the Fourier transform of  $h$ , that is, the CHFT just defined in expression (38). The convolution theorem applied to expression (41) renders:

$$g(x) = \int_{-\infty}^{\infty} h(y) w(x-y) dy, \quad (42)$$

which is equivalent to:

$$g(x) = \int_{-\infty}^{\infty} f(y) \exp(2\pi jP(y)) w(x-y) dy. \quad (43)$$

A simple numerical scheme to compute expression (43) offers an interesting application of Lemma 1. In fact, if we approximate that integral by the Riemann sum:

$$g(x) \approx \sum_{n=0}^N [f(y_n) w(x - y_n) \Delta y] \exp(2\pi j P(y_n)), \quad (44)$$

we can, from a straightforward application of Lemma 1, estimate the  $f(x)$  through:

$$f(x) \approx \left( \sum_{n=0}^N [f(y_n) w(x - y_n) \Delta y]^2 \right)^{1/2}. \quad (45)$$

This expression is used in [1] to check the quality of the recovered signal.

#### 4.1 CHFT Analysis

The first point to be demonstrated is the fact that we can recover the input signal  $f(x)$  if we take the inverse Fourier transform of the CHT given by:

$$F^{-1}(H(\omega)) = \int_{-\infty}^{\infty} H(\omega) \exp(2\pi j \omega x) d\omega. \quad (46)$$

If we substitute the CHT  $H(\omega)$  by expression (38) we obtain:

$$F^{-1}(H(\omega)) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(y) \exp(-2\pi j \omega y) \exp(2\pi j P(y)) dy \right] \exp(2\pi j \omega x) d\omega. \quad (47)$$

Using development of [9] we shall introduce the kernels  $\exp(-\omega^2/n^2)$  in order to deal with convergence problems for this integral. Therefore, we get:

$$f_n(x) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(y) \exp(-2\pi j \omega y) \exp(2\pi j P(y)) dy \right] \exp(2\pi j \omega x) \exp\left(-\frac{\omega^2}{n^2}\right) d\omega. \quad (48)$$

Now, we can change the order of integration and re-write this expression as:

$$f_n(x) = \int_{-\infty}^{\infty} f(y) \exp(2\pi j P(y)) \left[ \int_{-\infty}^{\infty} \exp[2\pi j \omega (x - y)] \exp\left(-\frac{\omega^2}{n^2}\right) d\omega \right] dy. \quad (49)$$

The inner integral has analytical solution named by  $k_n(x - y)$ :

$$k_n(x - y) = \int_{-\infty}^{\infty} \exp[2\pi j \omega (x - y)] \exp\left(-\frac{\omega^2}{n^2}\right) d\omega, \quad (50)$$

which gives:

$$k_n(x - y) = \int_{-\infty}^{\infty} \cos[2\pi j \omega (x - y)] \exp\left(-\frac{\omega^2}{n^2}\right) d\omega + \quad (51)$$

$$j \int_{-\infty}^{\infty} \sin[2\pi j \omega (x - y)] \exp\left(-\frac{\omega^2}{n^2}\right) d\omega. \quad (52)$$

From the fact that *sine* is an odd function and *exp* is an even one, it is easy to shown that the imaginary part will be zero and the real part of the last expression can be computed as:

$$k_n(x - y) = 2 \int_0^{\infty} \cos[2\pi \omega (x - y)] \exp\left(-\frac{\omega^2}{n^2}\right) d\omega. \quad (53)$$

This integral has analytical solution that can be obtained by expression [10]:

$$\int_0^\infty \cos bx \exp(-ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{-b^2}{4a}\right). \quad (54)$$

Henceforth, the application of this expression gives:

$$k_n(x-y) = n\sqrt{\pi} \exp\left[-\frac{[2\pi(x-y)]^2 n^2}{4}\right]. \quad (55)$$

Therefore, returning to equation (49) we obtain:

$$f_n(x) = \int_{-\infty}^\infty f(y) \exp(2\pi j P(y)) k_n(x-y) dy. \quad (56)$$

It can be proved that (see [9] for details):

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \exp(2\pi j P(x)). \quad (57)$$

But, we have also:

$$f_n(x) = \int_{-\infty}^\infty H(\omega) \exp(2\pi j \omega x) \exp\left(-\frac{\omega^2}{n^2}\right) d\omega. \quad (58)$$

It is also possible to prove that:

$$\lim_{n \rightarrow \infty} \int_{-\infty}^\infty H(\omega) \exp(2\pi j \omega x) \exp\left(-\frac{\omega^2}{n^2}\right) d\omega = \int_{-\infty}^\infty H(\omega) \exp(2\pi j \omega x) d\omega. \quad (59)$$

From (57) and (59) it follows that:

$$f(x) \exp(2\pi j P(x)) = \int_{-\infty}^\infty H(\omega) \exp(2\pi j \omega x) d\omega, \quad (60)$$

and so, as expected the whole signal is recovered as the amplitude of the inverse Fourier transform of the CHT.

## 5 Progressive Transmission

In this section we propose a progressive transmission approach based on the DHFT. This is performed by dividing the image (in the holographic domain) in several equally sized portions. Then, we take the recovered signal of a portion and incrementally add those ones corresponding to the other pieces.

Formally, given  $W_1$  and  $W_2$  two window functions similar to expression (15), we have:

$$I_{w_1}(r) = \sum_{u=0}^{M-1} H(u) W_1(u) \frac{1}{\sqrt{M}} e^{-j \frac{2\pi}{M} ur}, \quad (61)$$

$$I_{w_2}(r) = \sum_{u=0}^{M-1} H(u) W_2(u) \frac{1}{\sqrt{M}} e^{-j \frac{2\pi}{M} ur}, \quad (62)$$

where  $H(u)$  is the DHFT defined in equation (13). Thus, the linearity of the Fourier transform allows to write:

$$I_{w_1}(r) + I_{w_2}(r) = \sum_{u=0}^{M-1} H(u) (W_1(u) + W_2(u)) \frac{1}{\sqrt{M}} e^{-j \frac{2\pi}{M} ur}, \quad (63)$$

and so:

$$I_{w_1}(r) + I_{w_2}(r) = I_{w_1+w_2}(r). \quad (64)$$

Observe that if  $W_1, W_2$  are disjoint windows,  $W_1 + W_2$  is another window function. Consequently, there is not changes in the spectrum of the signal inside the window.

Thus, following expression (61)-(62), each recovered portion ( $I_{w_1}(r)$  and  $I_{w_2}(r)$  above) can be considered as a packet to be transmitted, whose arrival order does not matter.

## 6 Conclusions

In this paper we discuss some results concerning to Fourier analysis and Holographic representation of images.

The approach described in section 5 must be analyzed from the viewpoint of progressive transmission approaches [14, 6]. Certainly, the need of double precision representation and the known instability of Fourier analysis [7] may be several limitations for such method, if compared with other ones [6, 14]. Probably, the main areas of application for the Holographic transform would be in distributed words and watermarking methods. Besides a more complete theoretical analysis for CHFT must offered in order to quantify how good the estimate in expression (43) is for the input signal  $f$ .

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## 7 Appendix

In this section we review interesting properties in discrete Fourier analysis.

**Property A1:** The circular convolution theorem states that the DFT of the circular convolution of two sequences is equal to the product of their DFTs, that is, if:

$$x_2(n) = \sum_{k=0}^{N-1} h(n-k)_c x_1(k), \quad 0 \leq n \leq 1, \quad (65)$$

where  $h(n-k)_c$  is the a circular kernel (see [16]) then:

$$DFT[x_2(n)] = DFT[h(n)] DFT[x_1(n)]. \quad (66)$$

Expression (66) can be written as:

$$\sum_{k=0}^{N-1} x_2(k) e^{-j\frac{2\pi}{N}vk} = \sum_{k=0}^{N-1} h(k) e^{-j\frac{2\pi}{N}vk} \sum_{k=0}^{N-1} x_1(k) e^{-j\frac{2\pi}{N}vk}.$$

If we divide this expression by  $\sqrt{N}$  we get:

$$\sum_{k=0}^{N-1} x_2(k) \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}vk} = \sum_{k=0}^{N-1} h(k) \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}vk} \sum_{k=0}^{N-1} x_1(k) e^{-j\frac{2\pi}{N}vk}.$$

Therefore, we get the same relation for the Unitary Discrete Fourier Transform:

$$UDFT[x_2(n)] = UDFT[h(n)] DFT[x_1(n)]. \quad (67)$$

Now, if we define

$$x_1(k) = \begin{cases} png1, & k = a, \\ 0, & k \neq a, \end{cases}$$

we have that, from expression (65),  $x_2(n) = h(n-a)_c$ , i.e.,  $h(n)$  is a circular shift of  $x_2(n)$ . Hence

$$UDFT[x_2(n)] = UDFT[h(n)] \sum_{k=0}^{N-1} x_1(k) e^{-j\frac{2\pi}{N}vk},$$

which implies

$$UDFT[x_2(n)] = UDFT[h(n)] e^{-j\frac{2\pi}{N}va}.$$

**Property A2.** Let us consider the summation:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = \sum_{u=0}^{L-1} (e^{-j2\pi x})^u. \quad (68)$$

By remembering the geometric series:

$$S = \sum_{u=0}^{L-1} a^u = 1 + a + a^2 + \dots + a^{L-1},$$

we know that:

$$S = \frac{1 - a^L}{1 - a}.$$

If we apply this result in expression (68) we get:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = \frac{1 - (e^{-j2\pi x})^L}{1 - e^{-j2\pi x}} = \frac{1 - e^{-j2\pi xL}}{1 - e^{-j2\pi x}} = \frac{e^{-j\pi xL} (e^{j\pi xL} - e^{-j\pi xL})}{e^{-j\pi x} (e^{j\pi x} - e^{-j\pi x})}, \quad (69)$$

But we know that:

$$\sin(\pi x) = \frac{e^{j\pi x} - e^{-j\pi x}}{2j},$$

$$\sin(L\pi x) = \frac{e^{j\pi xL} - e^{-j\pi xL}}{2j}.$$

Therefore, expression (69) becomes:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = \frac{e^{-j\pi xL} 2j \sin(L\pi x)}{e^{-j\pi x} 2j \sin(\pi x)} = e^{-j(L-1)\pi x} \frac{\sin(L\pi x)}{\sin(\pi x)}. \quad (70)$$

Hence, if  $x = 0$  we get:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = L.$$

We can also re-write expression (70) by using the *sinc* function, which is defined by:  $\text{sinc}(x) = \frac{\sin(x)}{x}$ . Thus, that expression becomes:

$$\sum_{u=0}^{L-1} e^{-j2\pi xu} = e^{-j(L-1)\pi x} \frac{L\pi x \cdot \text{sinc}(L\pi x)}{\pi x \cdot \text{sinc}(\pi x)} = L e^{-j(L-1)\pi x} \frac{\text{sinc}(L\pi x)}{\text{sinc}(\pi x)}.$$